16.1

Double Integrals over Rectangular Regions

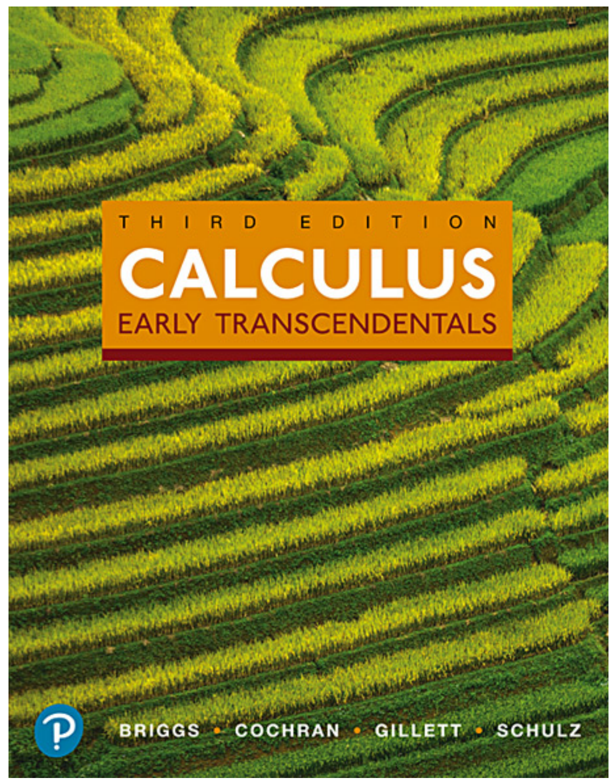
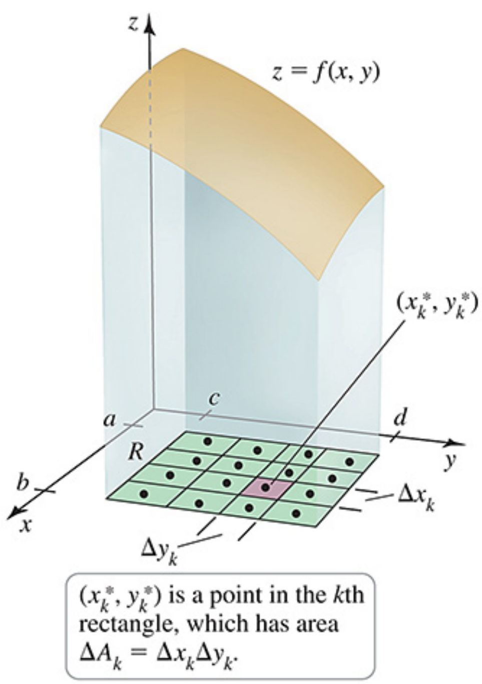


Table 16.1

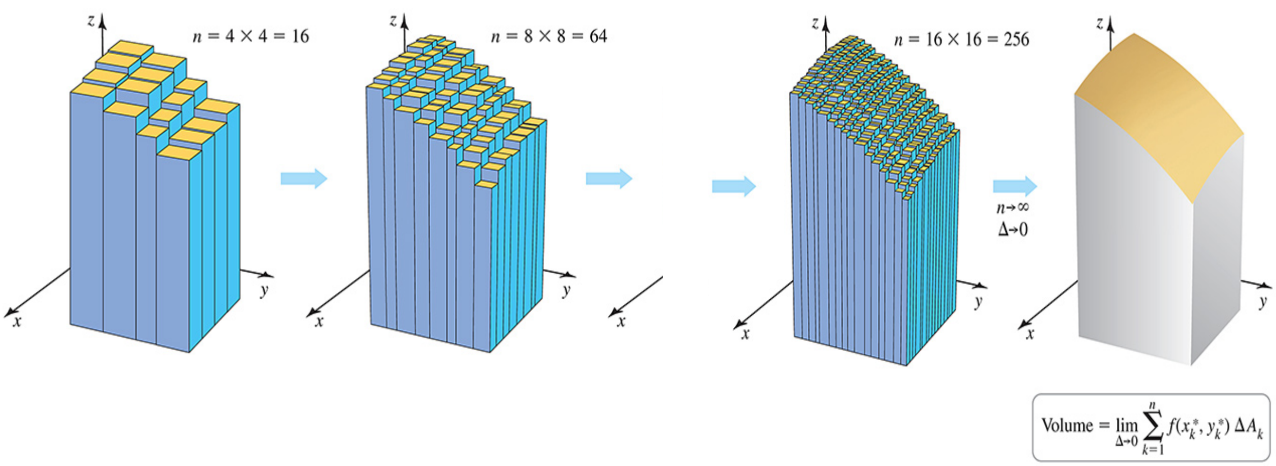
	Derivatives	**Integrals**
Single variable: $f(x)$	$f'(x)$	$\int_a^b f(x)\,dx$
Several variables: $f(x, y)$ and $f(x, y, z)$	$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$	$\iint_R f(x, y)\,dA$, $\iiint_D f(x, y, z)\,dV$

Figure 16.2



z
$z = f(x, y)$
(x_k^*, y_k^*)
a
c
d
R
y
b
x
Δx_k
Δy_k
(x_k^*, y_k^*) is a point in the kth rectangle, which has area $\Delta A_k = \Delta x_k \Delta y_k$.
z
$z = f(x, y)$
Height of kth box $= f(x_k^*, y_k^*)$
a
c
d
b
x
y
(x_k^*, y_k^*)
Volume of kth box $= f(x_k^*, y_k^*)\,\Delta A_k$

Figure 16.4 (2 of 2)



Δ = max length of diagonal.

DEFINITION Double Integrals

A function f defined on a rectangular region R in the xy-plane is **integrable** on R if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$ exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R**, which we write

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k.$$

THEOREM 16.1 **(Fubini) Double Integrals over Rectangular Regions**

Let f be continuous on the rectangular region $R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

$\int_c^d \int_a^b f(x.y)\,dx\,dy$: Integrate with respect to x then y.

$\int_a^b \int_c^b f(x.y)\,dy\,dx$: Integrate with respect to y then ~~x~~ x.

Example - Evaluate the double integral

$$\int_1^2 \int_0^1 (3x^2 + 4y^3)\ dy\,dx$$

$$\underbrace{\int_1^2 \int_0^1 (3x^2+4y^3)\,dy}_{\text{wrst } y \text{ first}}\,dx = \int_1^2 (3x^2y + y^4)\Big|_{y=0}^{y=1} dx$$

$$= \int_1^2 [(3x^2+1) - (0+0)]\,dx$$

$$= \int_1^2 (3x^2+1)\,dx$$

$$= (x^3+x)\Big|_{x=1}^{x=2}$$

$$= (2^3+2) - (1^3+1) = 8$$

Example - Evaluate the double integral

$$\int_1^2 \int_0^1 (3x^2 + 4y^3)\ dy\, dx$$

Reverse order (Fubini)

$$\int_0^1 \underbrace{\int_1^2 (3x^2+4y^3)\,dx}_{\text{w.r.t. } x \text{ first}}\, dy$$

$$= \int_0^1 (x^3 + 4y^3 x)\Big|_{x=1}^{x=2} dy$$

$$= \int_0^1 [(8+8y^3)-(1+4y^3)]\,dy$$

$$= \int_0^1 (7+4y^3)\,dy.$$

$$= (7y+y^4)\Big|_{y=0}^{y=1}$$

$$= (7+1)-(0+0) = 8.$$

Example - Evaluate double integral

$$\iint_R \frac{x}{1+xy}\, dA \qquad R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$$

$$\int \frac{1}{1+ay}\,dy = \frac{1}{a}\ln|1+ay| + C.$$

$$\int_0^1 \boxed{\int_0^1 \frac{x}{1+xy}\,dy}\,dx = \int_0^1 \frac{x}{x}\ln|1+xy|\Big|_0^1\,dx.$$

$$= \int_0^1 \ln|1+x| - \ln|1| \qquad dx$$

$$= \int_0^1 \ln|1+x|\,dx$$

$$\int \ln u\,du = u\ln u - u + C$$

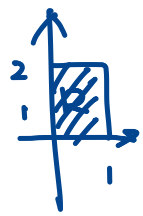
$$= \int_0^1 \ln(1+x)\,dx$$

$$= (1+x)\ln(1+x) - (1+x)\Big|_0^1$$

$$= (2\ln(2) - 2) - (\ln(1) - 1) = 2\ln(2) - 1 = \ln 4 - 1$$

Example - Find the volume

The solid in the first octant bounded above by the surface
$z = 9xy\sqrt{1-x^2}\sqrt{4-y^2}$



$$z=0 \Rightarrow 9xy\sqrt{1-x^2}\sqrt{4-y^2}=0 \Rightarrow x=0,1\ ;\ y=0.2.$$

$$\iint_R 9xy\sqrt{1-x^2}\sqrt{4-y^2}\,dx\,dy = \int_0^2\int_0^1 9xy\sqrt{1-x^2}\sqrt{4-y^2}\,dx\,dy$$

$$= \int_0^2\int_0^1 \underbrace{\left(9y\sqrt{4-y^2}\right)}_{\text{constant}} \underbrace{x\sqrt{1-x^2}}\,dx\,dy$$

$u = 1-x^2$

$-\frac{1}{2}du = x\,dx$

$$= \int_0^2 9y\sqrt{4-y^2}\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(1-x^2)^{\frac{3}{2}}\Big|_0^1 dy$$

$$= -3\int_0^2 y\sqrt{4-y^2}\left(0^{3/2}-1^{3/2}\right)dy$$

$u = 4-y^2$

$-\frac{1}{2}du = y\,dy$

$$= -\frac{3}{2}\cdot\frac{2}{3}(4-y^2)^{\frac{3}{2}}\Big|_0^2$$

$$= -\left[0^{3/2}-4^{\frac{3}{2}}\right]$$

$$= 8$$

Example - Choose a convenient order

$$\iint_R y\cos xy \; dA \qquad R = \left\{(x,y) : 0 \le x \le 1, 0 \le y \le \frac{\pi}{3}\right\}$$

$$\underbrace{\int_0^{\frac{\pi}{3}}\int_0^1 y\cos xy \, dx\,dy}_{\text{use basic integral.}} = \int_0^{\frac{\pi}{3}} \qquad \text{or} \qquad \int_0^1 \underbrace{\int_0^{\frac{\pi}{3}} y\cos(xy)\,dy}_{\text{Integration by parts.}}\,dx.$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C.$$

$u = y \qquad dv = \cos(xy)\,dy$

$du = dy \qquad v = \dfrac{\sin(xy)}{x}$

Example - Choose a convenient order

$$\iint_R y\cos xy\ dA \qquad R = \left\{(x,y) : 0 \le x \le 1, 0 \le y \le \frac{\pi}{3}\right\}$$

$$\int_0^{\frac{\pi}{2}}\int_0^1 y\cos(xy)\,dx\,dy = \int_0^{\frac{\pi}{2}} \frac{y}{y}\sin(xy)\Big|_0^1 dy.$$

$$= \int_0^{\frac{\pi}{2}} \sin(y) - \sin(0)\,dy$$

$$= -\cos(y)\Big|_0^{\pi/3}$$

$$= -\cos\frac{\pi}{3} + \cos 0$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}.$$

DEFINITION Average Value of a Function over a Plane Region

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA.$$

Average height of $f(x,y) = \frac{1}{\text{Area}(R)}$ (Volume under $f(x,y)$).

Example - Average value

Compute the average value of f over the region R.

$f(x, y) = 4 - x - y; \qquad R = \{(x, y) : 0 \le x \le 2, 0 \le y \le 2\}$

$$\frac{1}{\text{Area}(R)} \iint_R (4-x-y)\,dA = \frac{1}{4}\int_0^2\int_0^2 (4-x-y)\,dx\,dy$$

Area of R $2 \cdot 2 = 4$

$$= \frac{1}{4}\int_0^2 \left(4x - \frac{x^2}{2} - xy\right)\Big|_0^2 dy$$

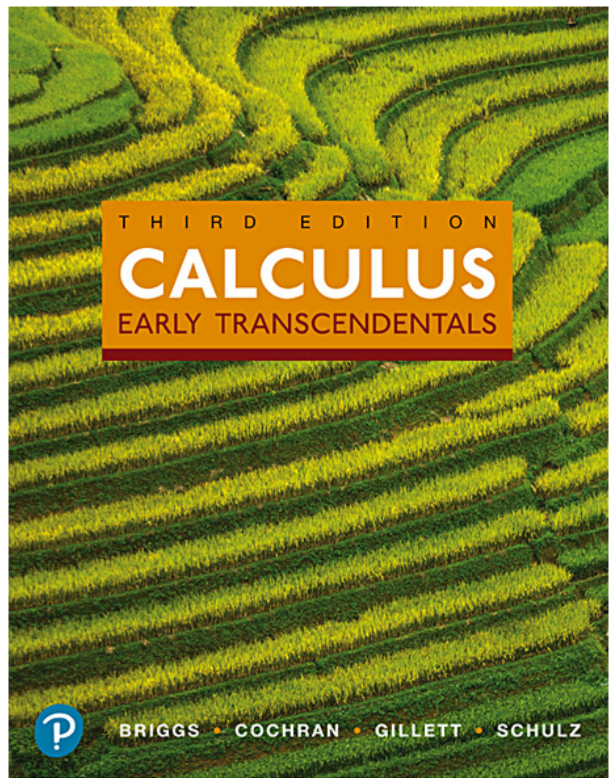
$$= \frac{1}{4}\int_0^2 (8 - 2 - 2y)\,dy$$

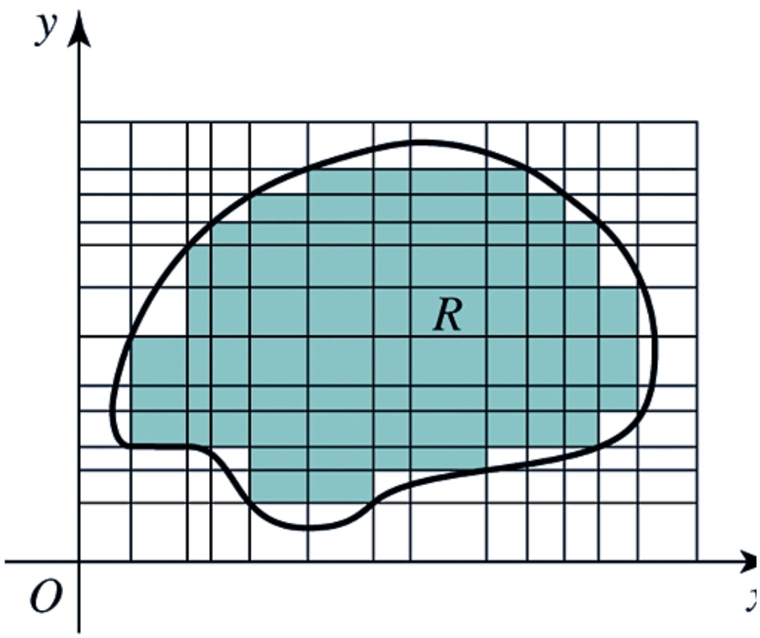
$$= \frac{1}{4}\int_0^2 (6 - 2y)\,dy$$

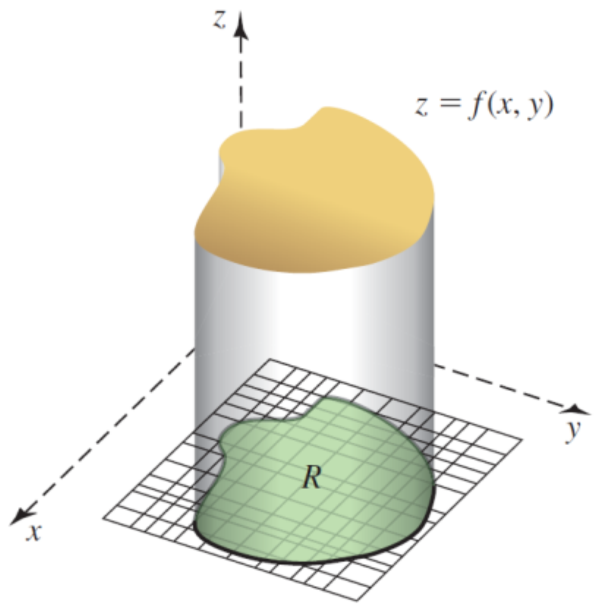
$$= \frac{1}{4}(6y - y^2)\Big|_0^2 = \frac{1}{4}(12 - 4) = 2.$$

16.2

Double Integrals over General Regions





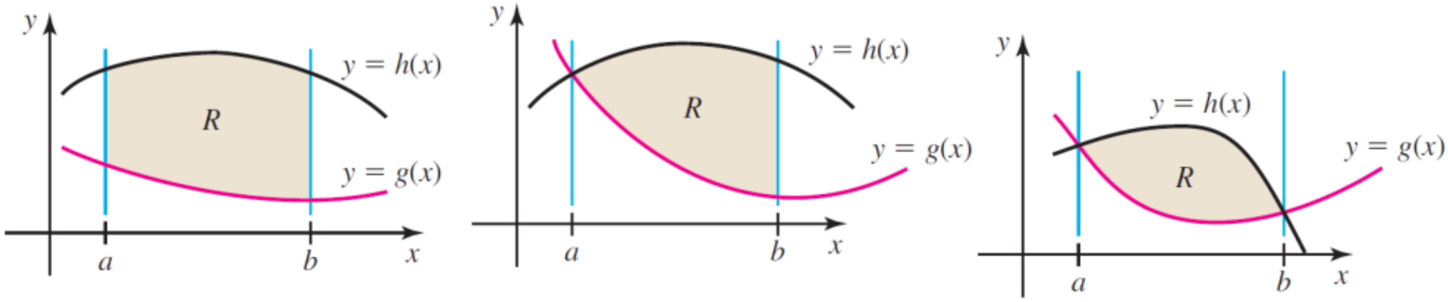


$$\text{Volume of solid} = \iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$

iterated Integrals

dydx

Figure 16.11

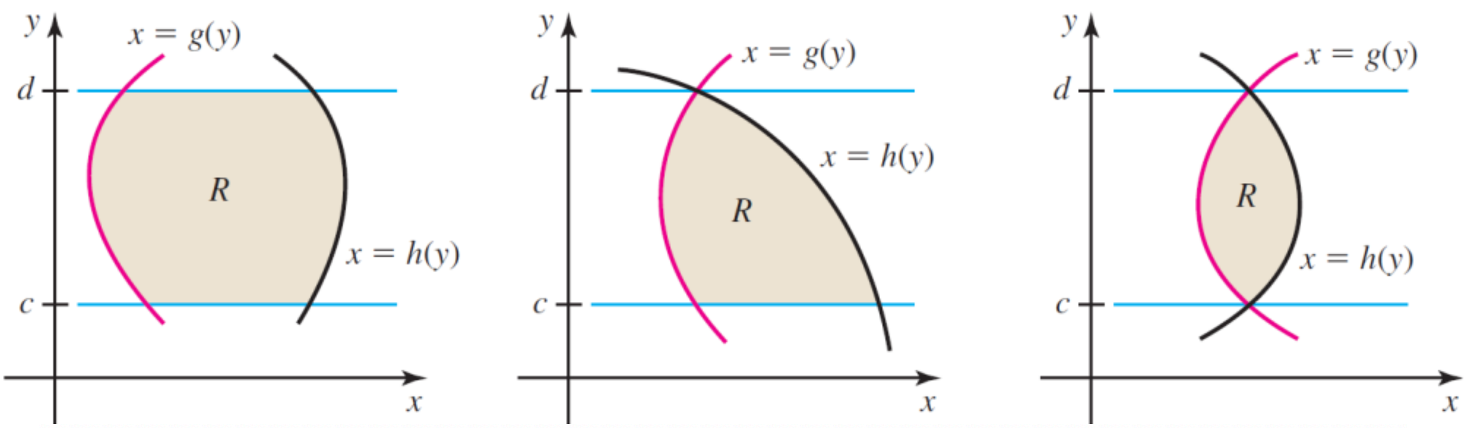


THEOREM 16.2 Double Integrals over Nonrectangular Regions
Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$ (Figure 16.11). If f is continuous on R, then

$$\iint_R f(x, y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\,dy\,dx.$$

dxdy

Figure 16.15



Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$ (Figure 16.15). If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

Example - Evaluate the double integral

$$\int_0^1 \underbrace{\int_0^x 2e^{x^2}\, dy}\, dx = \int_0^1 2e^{x^2} y \Big|_0^x dx$$

Integrate with respect to y

$$= \int_0^1 (2xe^{x^2} - 0)\, dx$$

$$\boxed{\begin{aligned} &\text{Let } u = x^2 \\ &du = 2x\,dx \end{aligned}}$$

$$= \int_0^1 e^u\, du$$

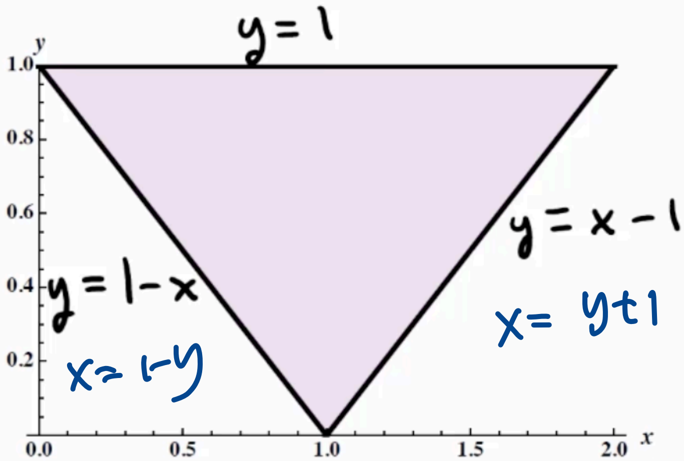
$$= e^u \Big|_0^1$$

$$= e^1 - e^0 = e - 1$$

Example - Evaluate the double integral

$$\iint_R y^2\, dA$$

R is the region bounded by $y = 1$, $y = 1 - x$ and $y = x - 1$.



$$\int_0^1 \int_{1-y}^{y+1} y^2\, dx\, dy.$$

$$= \int_0^1 y^2 x \Big|_{1-y}^{y+1} dy.$$

$$= \int_0^1 \left[y^2(1+y) - y^2(1-y) \right] dy$$

$$= \int_0^1 2y^3\, dy$$

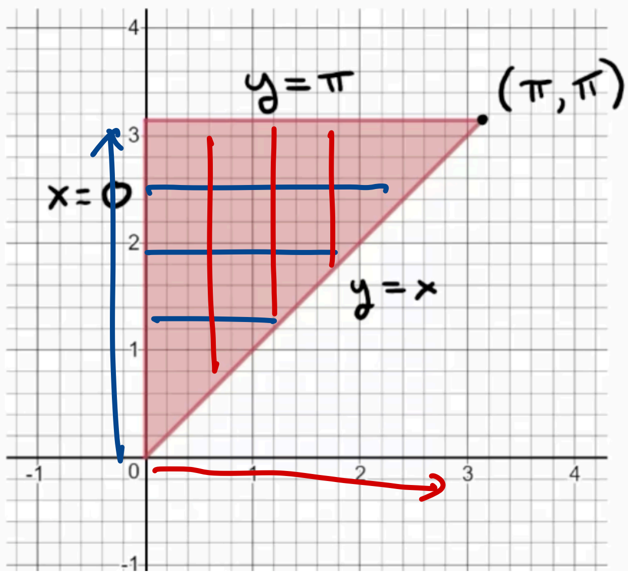
$$= \frac{1}{2} y^4 \Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Example - Evaluate the double integral

$\iint\limits_R y^2 \, dA$ R is the region bounded by $y = 1$, $y = 1 - x$ and $y = x - 1$.

Example - Find the volume

Find the volume of the solid bounded between the cylinder $z = \sin^2 x$ and the xy-plane over the region $R = \{(x, y): 0 \le x \le y \le \pi\}$



$$\text{Volume} = \iint_R f(x,y)\,dA$$

$$\int_0^{\pi}\int_0^{y} \sin^2 x\,dx\,dy$$

OR.

$$\int_0^{\pi}\int_x^{\pi} \sin^2 x\,dy\,dx$$

Example - Find the volume

Find the volume of the solid bounded between the cylinder $z = \sin^2 x$ and the xy-plane over the region $R = \{(x, y): 0 \le x \le y \le \pi\}$

$$\int_0^{\pi}\int_0^{y} \sin^2 x \, dx\, dy = \int_0^{\pi} \frac{1}{2}\int_0^{y} (2\sin^2 x)\, dx\, dy$$

$$= \int_0^{\pi} \frac{1}{2}\int_0^{y} (1+\cos 2x)\, dx\, dy$$

$$= \int_0^{\pi} \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^{y} dy.$$

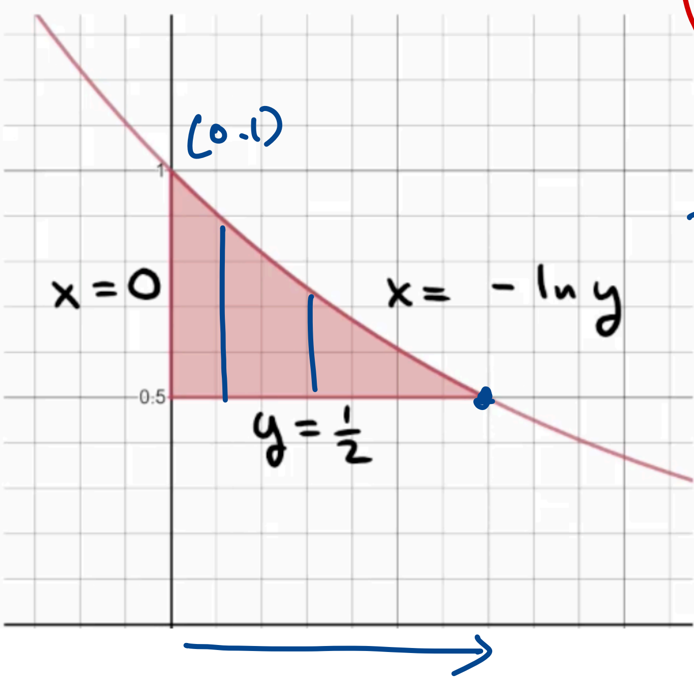
$$= \int_0^{\pi} \left(\frac{1}{2}y - \frac{1}{4}\sin(2y)\right) dy = \left[\frac{1}{4}y^2 + \frac{1}{8}\cos(2y)\right]\Big|_0^{\pi}$$

$$= \left(\frac{1}{4}\pi^2 + \frac{1}{8}\cos 2\pi\right) - \left(\frac{1}{4}0^2 + \frac{1}{8}\cos 0\right)$$

$$= \frac{\pi^2}{4}.$$

Example - Reverse the order of integration

$$\int_{\frac{1}{2}}^{1}\int_{0}^{-\ln y} f(x,y)\,dx\,dy = \boxed{\int_{0}^{\ln 2}\int_{\frac{1}{2}}^{e^{-x}} f(x,y)\,dy\,dx.}$$



$x = -\ln y$

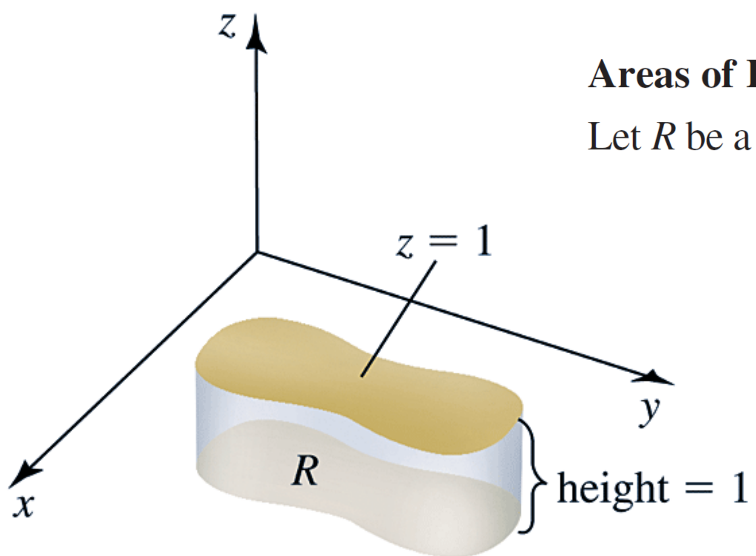
$-x = \ln y$

$e^{-x} = y.$

If $y = \frac{1}{2}.$

$x = -\ln\left(\frac{1}{2}\right)$

$= \ln 2$



Areas of Regions by Double Integrals

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint_R dA.$$

Area by double integral.

$$\text{Volume} = \iint_R 1\, dA$$

$$\text{Volume} = (1 \times \text{Are of } R)$$

$$= \text{Area of } R.$$

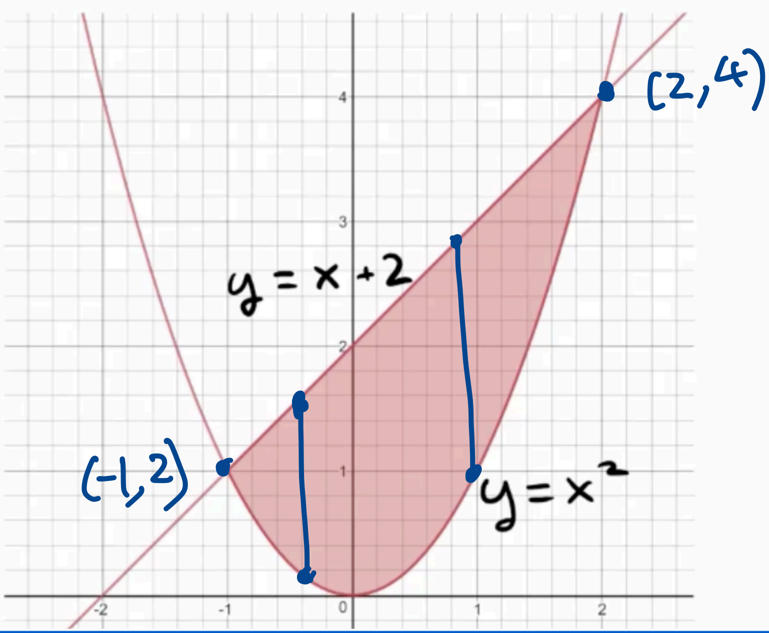
$\therefore$

$$\text{Area of} = \iint_R 1\, dA$$

Volume of solid = (area of R) $\times$ (height)
= area of R = $\iint_R 1\, dA$

Example - Find the area

The region bounded by the parabola $y = x^2$ and the line $y = x + 2$.



$$Area(R) = \iint_R 1\, dA$$

$$\int_{-1}^{2} \int_{x^2}^{x+1} 1\, dy \, . \, dx$$

$x^2 = x + 2.$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1, \; x = 2.$

Example - Find the area

The region bounded by the parabola $y = x^2$ and the line $y = x + 2$.

$$\text{Area} = \int_{-1}^{2} \int_{x^2}^{x+2} 1 \, dy \, dx$$

$$= \int_{-1}^{2} y \Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} [x+2 - x^2] \, dx$$

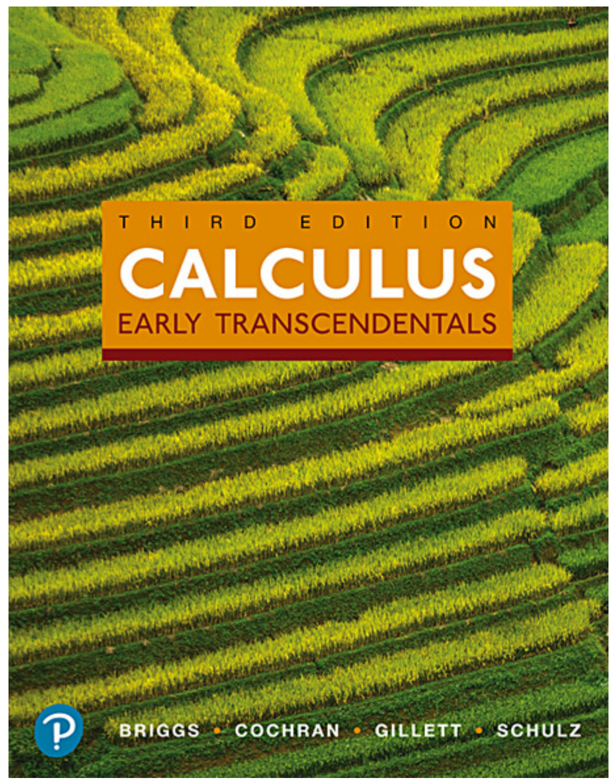
$$= \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

$$= \frac{1}{2}4 + 4 - \frac{8}{3} - \left[\frac{1}{2} - 2 + \frac{1}{3}\right] = 8 - \frac{8}{3} - \frac{1}{2} - \frac{1}{3} = \frac{15}{2} - \frac{6}{2}$$

$$= \frac{9}{2}$$

16.3

Double Integrals in Polar Coordinates



Cartesian Rectangle

$R = \{(x,y).\ a \le x \le b,\ c \le y \le d\}$

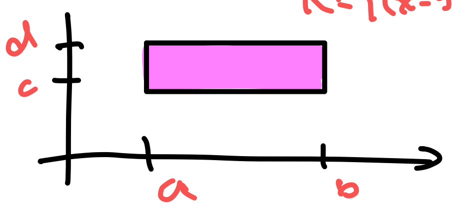


Figure 16.28

Polar Rectangle.

$R = \{(r,\theta): a \le r \le b,\ \alpha \le \theta \le \beta\}$

$x = r\cos\theta$

$y = r\sin\theta$

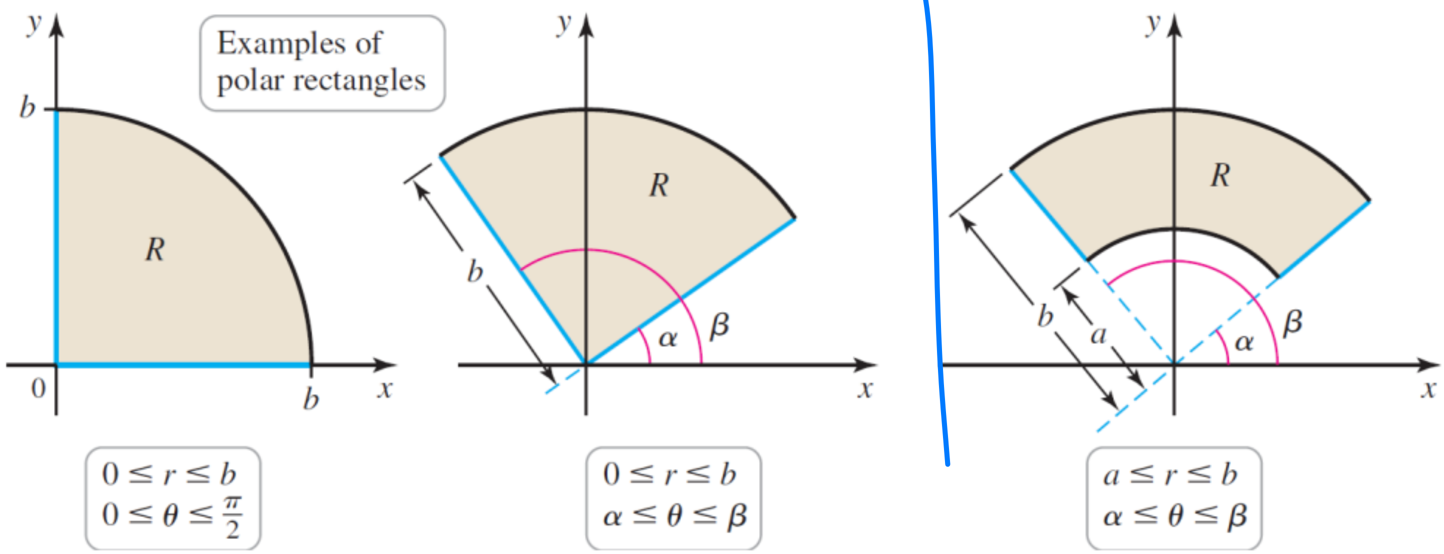


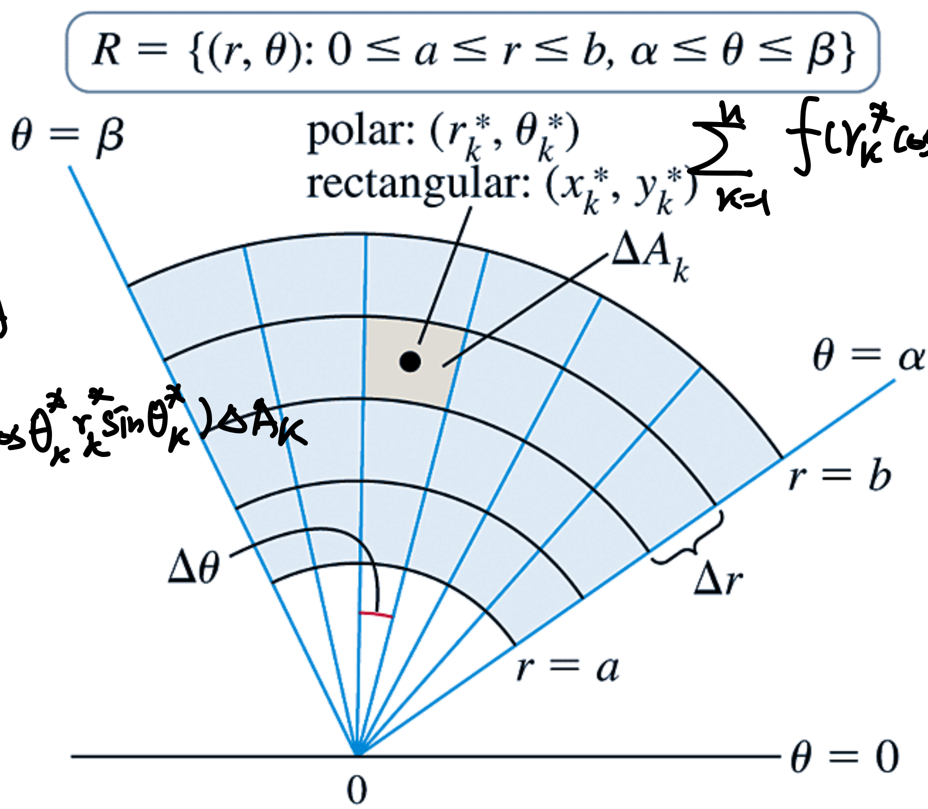
Figure 16.29

$f(x,y) = f(r\cos\theta, r\sin\theta)$

ΔA_k = Area of k-th polar rectangle.

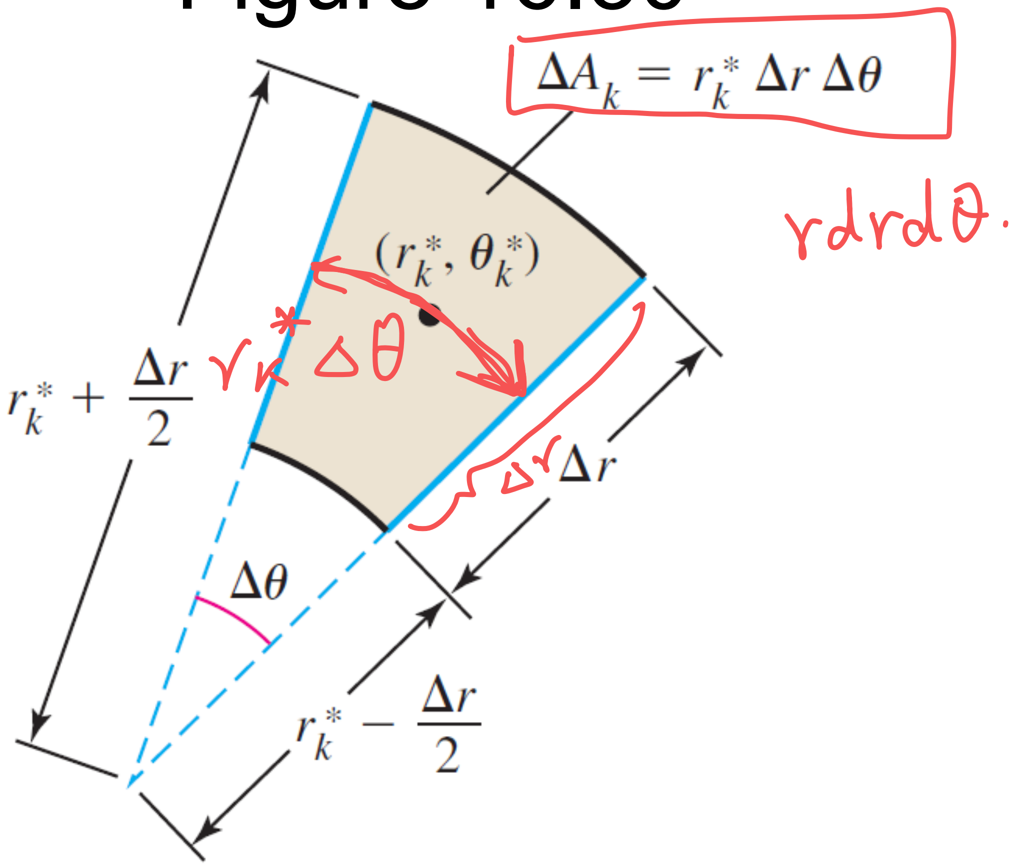
$\sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\Delta A_k$

$\iint_R f(x,y)\,dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\Delta A_k$



Iterated integrals in polar coordinates.

Figure 16.30



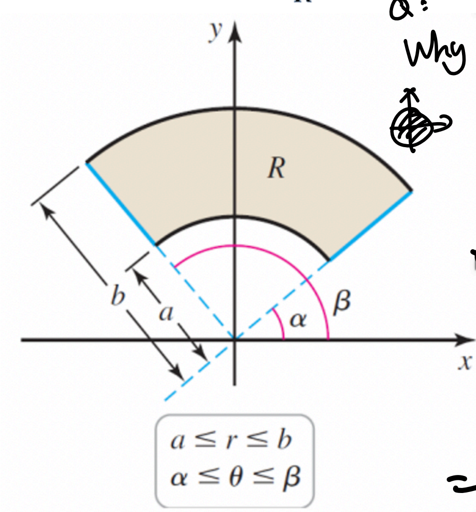
THEOREM 16.3 **Change of Variables for Double Integrals over Polar Rectangle Regions**

Let f be continuous on the region R in the xy-plane expressed in polar coordinates as $R = \{(r, \theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$. Then f is integrable over R, and the double integral of f over R is

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

$x = r\cos\theta$

$y = r\sin\theta$



Q:

Why are we doing this?

$\{(x,y) : -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ -1 \le x \le 1\}$ ← Rectangular

$\{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ ← Polar

$\iint_R x\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\, dy\, dx$

$= \int_0^{2\pi} \int_0^1 r\cos\theta \cdot r\, dr\, d\theta$

easier

e.g. Integrate $\iint_R 2xy\, dA$

$R = \{(r,\theta) : 1 \le r \le 3, 0 \le \theta \le \frac{\pi}{2}\}$

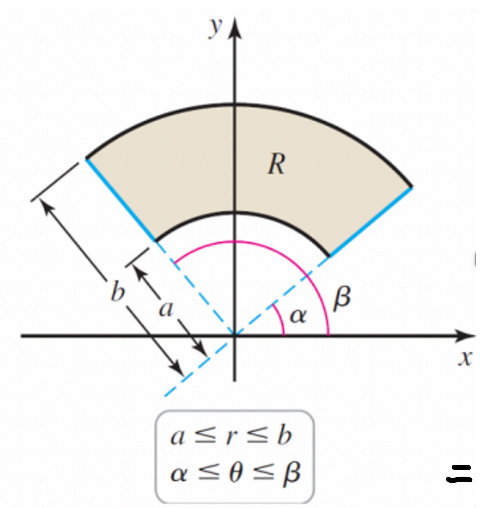
$\iint_R 2(r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}} \int_1^3 2r^3 \cos\theta \sin\theta\, dr\, d\theta$ $= 20$

THEOREM 16.3 Change of Variables for Double Integrals over Polar Rectangle Regions

Let f be continuous on the region R in the xy-plane expressed in polar coordinates as $R = \{(r, \theta): 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$. Then f is integrable over R, and the double integral of f over R is

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$



$$\int_0^{\pi/2}\int_1^3 2r^3\cos\theta\sin\theta\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{2}r^4\cos\theta\sin\theta\Big|_{r=1}^{3} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(3)^4\cos\theta\sin\theta - \frac{1}{2}(1)^4\cos\theta\sin\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{81}{2}\cos\theta\sin\theta - \frac{1}{2}\cos\theta\sin\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{80}{2}\cos\theta\sin\theta\, d\theta$$

$u = \sin\theta$ $\quad \theta = \frac{\pi}{2} \Rightarrow u = 1$

$du = \cos\theta\, d\theta$ $\quad \theta = 0 \Rightarrow u = 0$

$$= \int_0^1 40u\, du = 20u^2\Big|_0^1 = 20(1)^2 - 20(0)^2 = \boxed{20}$$

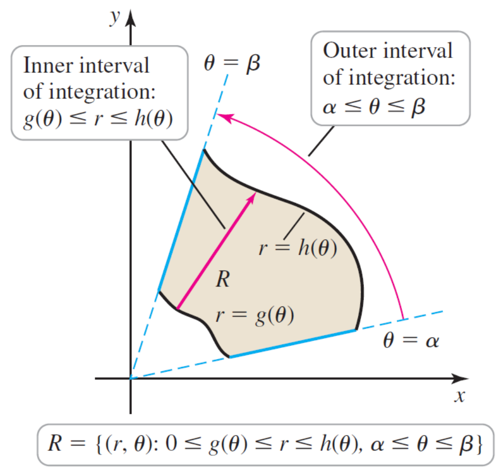
THEOREM 16.4 **Change of Variables for Double Integrals over More General Polar Regions**

Let f be continuous on the region R in the xy-plane expressed in polar coordinates as

$$R = \{(r, \theta): 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\},$$

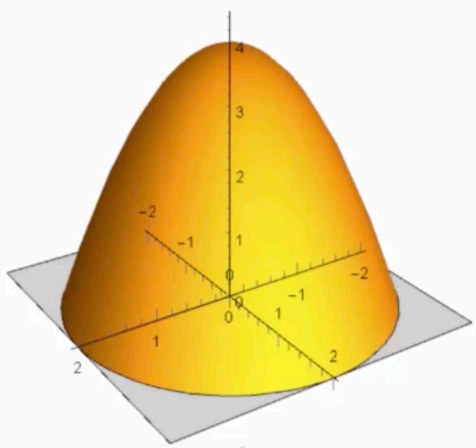
where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$



Example - Volume with polar coordinates

Find the volume of the solid bounded by the surface $z = 4 - x^2 - y^2$ and the xy-plane over the region $R = \{(r, \theta) : 0 \le r \le 2, 0 \le \theta \le 2\pi\}$



$$\text{Volume} = \iint_R f(x, y)\, dA$$

$$z = 4 - x^2 - y^2 = 4 - r^2\cos^2\theta - r^2\sin^2\theta = 4 - r^2$$

$$\iint_R f(x,y)\, dA = \int_0^{2\pi}\int_0^{2} (4 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 4r - r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} 2r^2 - \frac{1}{4}r^4 \Big|_0^2 d\theta$$

$$= \int_0^{2\pi} 8 - 4\, d\theta = 4\theta\Big|_0^{2\pi} = 8\pi$$

Region R.

Example - Cartesian to polar coordinates

Evaluate the integral over R using polar coordinates

$$\int_{-4}^{4} \boxed{\int_{0}^{\sqrt{16-y^2}} (16 - x^2 - y^2)\, dx}\, dy$$

Bounded on left by $x=0$

Bounded on right by

$x = \sqrt{16-y^2}$.

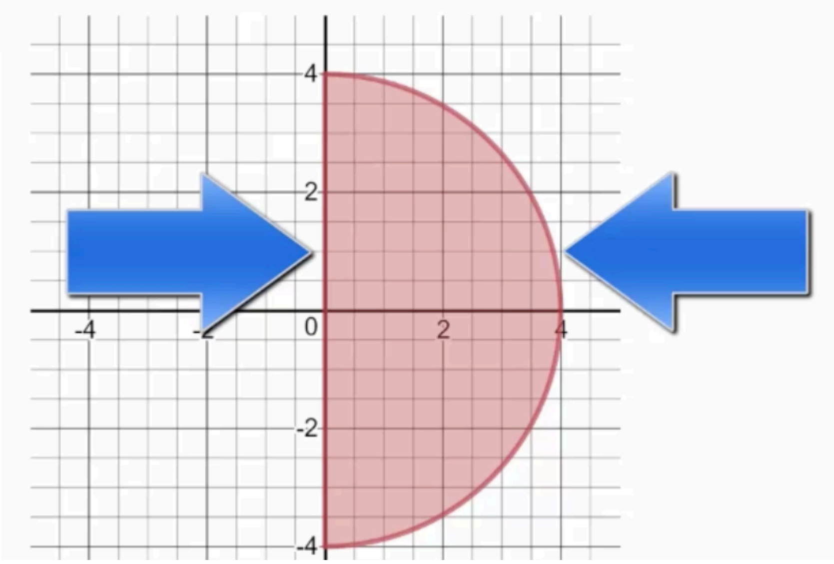
$x^2 = 16 - y^2$.

$x^2 + y^2 = 16 \quad x > 0$.

$r^2 = 16$

$r = 4$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 (16 - r^2)\, r\, dr\, d\theta$$



Example - Cartesian to polar coordinates

Evaluate the integral over R using polar coordinates

$$\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}} (16 - x^2 - y^2)\, dx\, dy$$

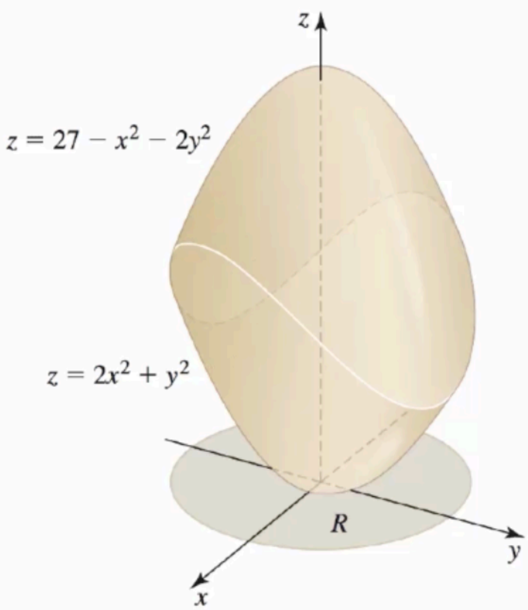
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{4} (16-r^2) r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{4} 16r - r^3\, dr\, d\theta.$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(8r^2 - \frac{1}{4}r^4\right)\Big|_0^4 d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\left(8\cdot 4^2 - \frac{1}{4}\cdot 4^4\right) - 0\right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 64\, d\theta = 64\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 64\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 64\pi$$

Example - Volume between surfaces

Find the volume of the solid bounded between the surfaces $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$.



Region R : $2x^2+y^2 = 27 - x^2 - 2y^2$.

$3x^2+3y^2 = 27$.

$3(x^2+y^2) = 27$

$x^2+y^2 = 9$.

R is a disk of radius 3.

centered at (0.0)

$$\text{Volume} = \iint_R [(27-x^2-2y^2) - (2x^2+y^2)]\,dA$$

$$= \iint_R 27 - 3x^2 - 3y^2\,dA = \iint_R 27 - 3(x^2+y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^3 (27-3r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 27r - 3r^3\,dr\,d\theta = \int_0^{2\pi} \frac{27}{2}r^2 - \frac{3}{4}r^4\Big|_0^3\,d\theta = \int_0^{2\pi}\frac{27}{2}3^2 - \frac{3}{4}3^4\,d\theta$$

$$= \int_0^{2\pi} \frac{243}{4}\,d\theta = \frac{243}{4}(2\pi - 0) = \frac{243\pi}{2}$$

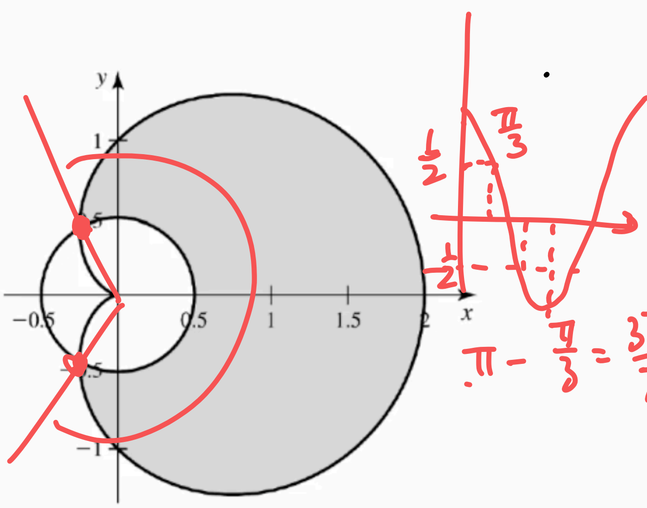
Region R : $\frac{1}{2} \le r \le 1+\cos\theta$

Bounded on θ : $\frac{1}{2} = 1+\cos\theta$

$-\frac{1}{2} = \cos\theta \quad -\frac{2\pi}{3} \le \theta \le \frac{2\pi}{3}$

Example - Describe a region

Write $\iint\limits_R f(r,\theta)\, dA$ as an iterated integral over R in polar coordinates where R is the region outside the circle $r = \frac{1}{2}$ and inside the cardioid $r = 1 + \cos\theta$.



$\pi - \frac{\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3}$

$$\iint\limits_R f(r,\theta)\, dA = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \int_{\frac{1}{2}}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$

How do we change $(x, y) \to (r, \theta)$?

$$\boxed{\iint_R f(x, y)\, dA.}$$

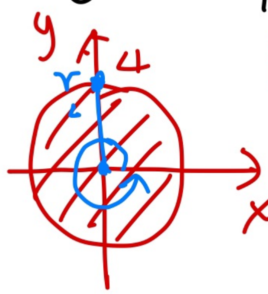
① rewrite $f(x, y)$. in terms of r, θ.

using $x = r\cos\theta$ $\quad y = r\sin\theta$. $\quad x^2 + y^2 = r^2$.

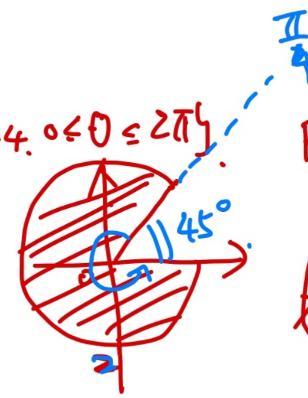
② change $dA = dx\,dy = dy\,dx \to \boxed{r\,dr\,d\theta} = dx\,dy$.

③. describe the region R. in terms of polar coordinates.

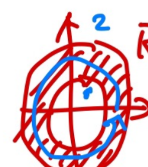
④ Integrate.



$R = \{(r, \theta) : 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

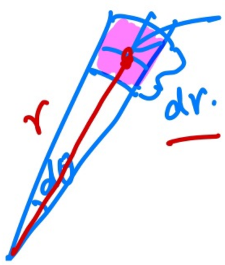


$R = \{(r, \theta) : 0 \le r \le 2,\ \frac{\pi}{4} \le \theta \le 2\pi\}$



$R = \{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$dA : r\,dr\,d\theta$



$r\,d\theta$

$r\,d\theta\,dr = dA.$

$x = r\cos\theta \qquad y = r\sin\theta.$

$$J(r.\theta) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = x_r y_\theta - x_\theta y_r$$

$$= \cos\theta \cdot (r\cos\theta) - (r - \sin\theta)\cdot(\sin\theta).$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\underbrace{\cos^2\theta + \sin^2\theta}_{1}). = r.$$

Definition Jacobian Determinant of a Transformation of Two Variables

Given a transformation $T:\ x = g(u, v),\ y = h(u, v),$ where g and h are differentiable on a region of the uv -plane, the **Jacobian determinant** (or **Jacobian**) of T is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Theorem 16.8 Change of Variables for Double Integrals

Let $T:\ x = g(u, v),\ y = h(u, v)$ be a transformation that maps a closed bounded region S in the uv -plane to a region R in the xy-plane. Assume T is one-to-one on the interior of S and g and h have continuous first partial derivatives there. If f is continuous on R, then

$r\theta.$

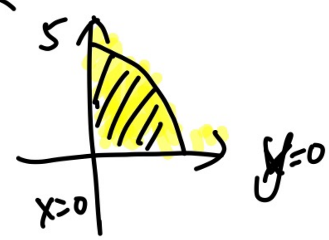
$$\iint_R f(x, y)\, dA = \iint_S f(g(u, v), h(u, v))\,|J(u, v)|\, dA.$$

$dxdy$

$|J(r.\theta)| = |r| = \boxed{r.}\, dr\,d\theta$

$$\iint_R \frac{1}{\sqrt{36-x^2-y^2}}\,dA \qquad R=\{(x,y):\ x^2+y^2\le 25.\ \ x\ge 0.\ y\ge 0\}$$

$x^2+y^2\le 5^2$.



$R=\{(r,\theta):\ 0\le r\le 5,\ \ 0\le\theta\le \frac{\pi}{2}\}$.

$x=r\cos\theta$ $y=r\sin\theta$. $x^2+y^2=r^2$.

$dA=r\,dr\,d\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^5 \frac{1}{\sqrt{36-r^2}}\,r\,dr\,d\theta. = \int_0^{\frac{\pi}{2}}\int_0^5 \frac{1}{\sqrt{36-r^2}}\,\frac{1}{2}dr^2\,d\theta$$

$36-r^2=u.$

$dr^2=-du.$

$r=0$ $u=36.$

$r=5$ $u=36-25$

$=11.$

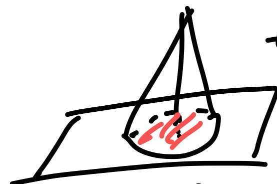
$$=\int_0^{\frac{\pi}{2}}\int_{36}^{11} \frac{1}{\sqrt{u}}\,\frac{-1}{2}\,du\,d\theta.$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_{36}^{11}\frac{1}{\sqrt{u}}\,du\,d\theta. = -\frac{1}{2}\int_0^{\frac{\pi}{2}} 2u^{\frac{1}{2}}\Big|_{36}^{11} d\theta. = -\int_0^{\frac{\pi}{2}}\sqrt{11}-6\,d\theta.$$

$$= -(\sqrt{11}-6)\cdot\frac{\pi}{2}. = -\sqrt{11}\,\frac{\pi}{2}+3\pi.$$

Homework Hints:

$f(x,y) = 25 - \sqrt{x^2+y^2}$



this cone is the surface of $f(x,y)$.

this is the region R.

How do we find region R:

this is in the x-y-plane $\to z=0$.

so let $25-\sqrt{x^2+y^2}=0 \to 25=\sqrt{x^2+y^2} \to 25^2=x^2+y^2$.

this means R is a circle of radius 25.

$$f(x,y) = e^{-\frac{(x^2+y^2)}{48}} - e^{-3}$$

Use the same technique set $e^{-\frac{(x^2+y^2)}{48}} - e^{-3} = 0$

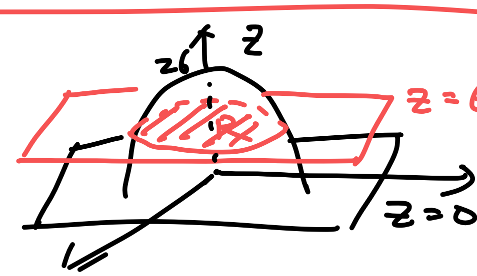
$$e^{-\frac{(x^2+y^2)}{48}} = e^{-3} \Rightarrow -\frac{(x^2+y^2)}{48} = -3 \Rightarrow x^2+y^2 = 3\cdot 48 \Rightarrow x^2+y^2=144$$

$\Rightarrow x^2+y^2=12^2 \Rightarrow$ R is circle of radius 12.

Solid bounded by $26-5x^2-5y^2$ and $z=6$.

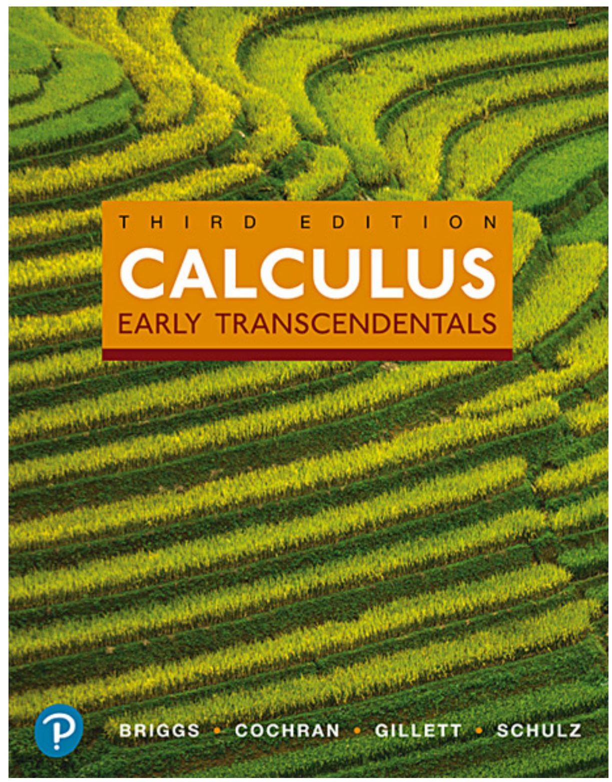
since R is on the plane $z=6$.

set $26-5x^2-5y^2=6$ and figure out where the region is. :)



16.4

Triple Integrals



Theorem 16.5 Triple Integrals

Let *f* be continuous over the region

$$D = \{(x, y, z) : \ a \le x \le b, g(x) \le y \le h(x), G(x, y) \le z \le H(x, y)\},$$

where *g*, *h*, *G*, and *H* are continuous functions. Then *f* is integrable over *D* and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

Notice the analogy between double and triple integrals:

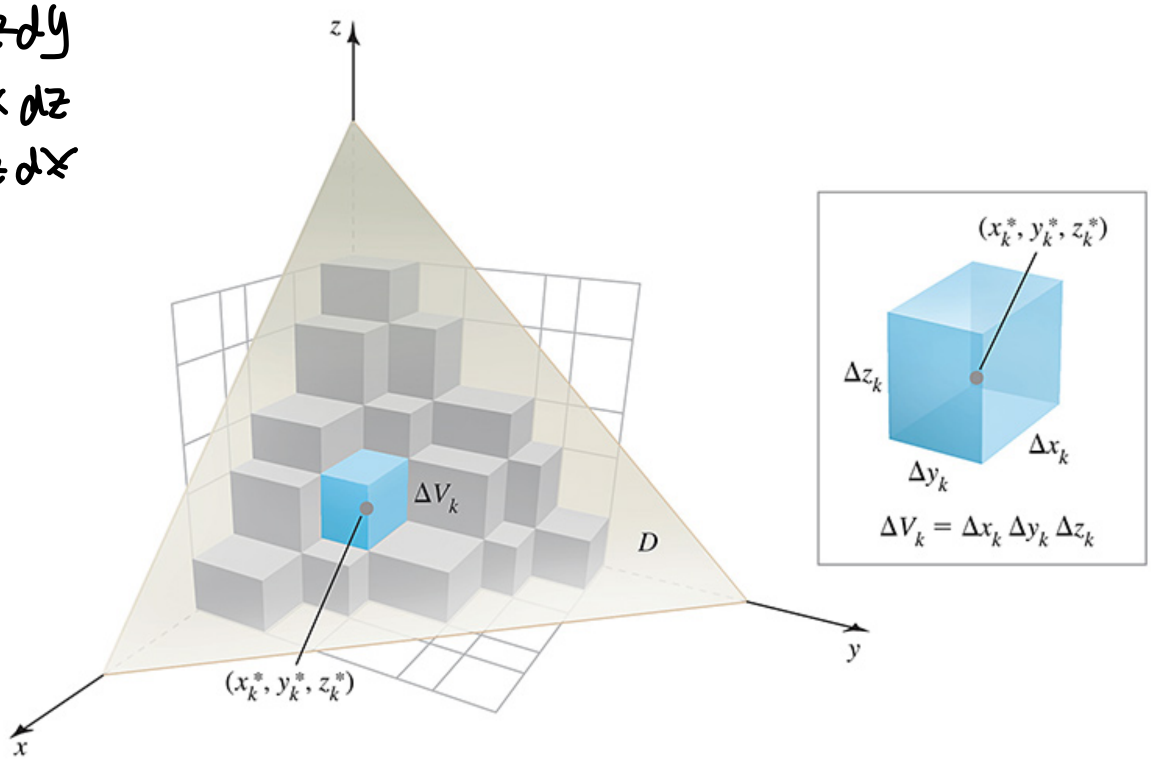
$$\text{area of } R = \iint_R dA \quad \text{and} \rightarrow \iint_R 1\,dA$$

$$\text{volume of } D = \iiint_D dV. \rightarrow \iiint_D 1\,dV$$

The use of triple integrals to compute the mass of an object is discussed in detail in Section 16.6.

dzdxdy
dzdydx
dxdydz
dxdzdy
dydxdz
dydzdx

Figure 16.38



Triple integrals

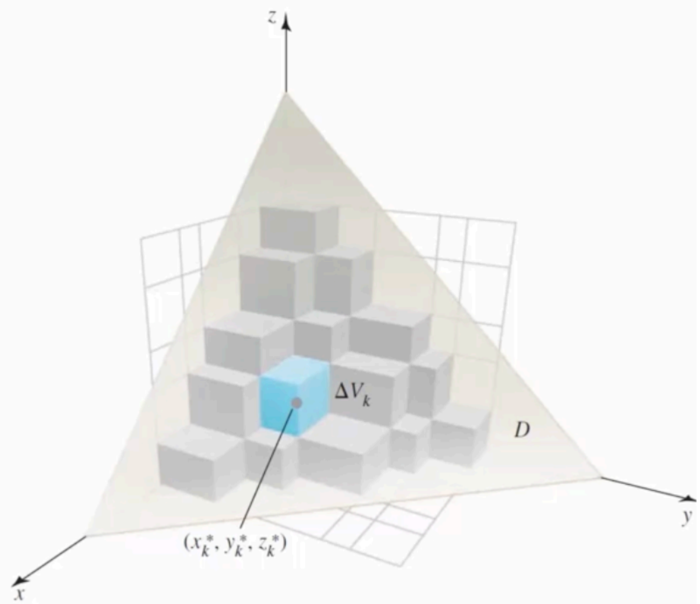
Two applications of triple integrals

1)

$$\iiint_D 1\, dV = \text{Volume}(D)$$

2) If $\rho(x, y, z)$ represents the density of a solid at any point (x, y, z) of a solid then

$$\iiint_D \rho(x, y, z)\, dV = \text{mass}(D)$$



THEOREM 16.5 Triple Integrals

Let f be continuous over the region

$$D = \{(x, y, z): a \le x \le b, g(x) \le y \le h(x), G(x, y) \le z \le H(x, y)\},$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x, y)}^{H(x, y)} f(x, y, z)\, dz\, dy\, dx.$$

x, y

x

constant

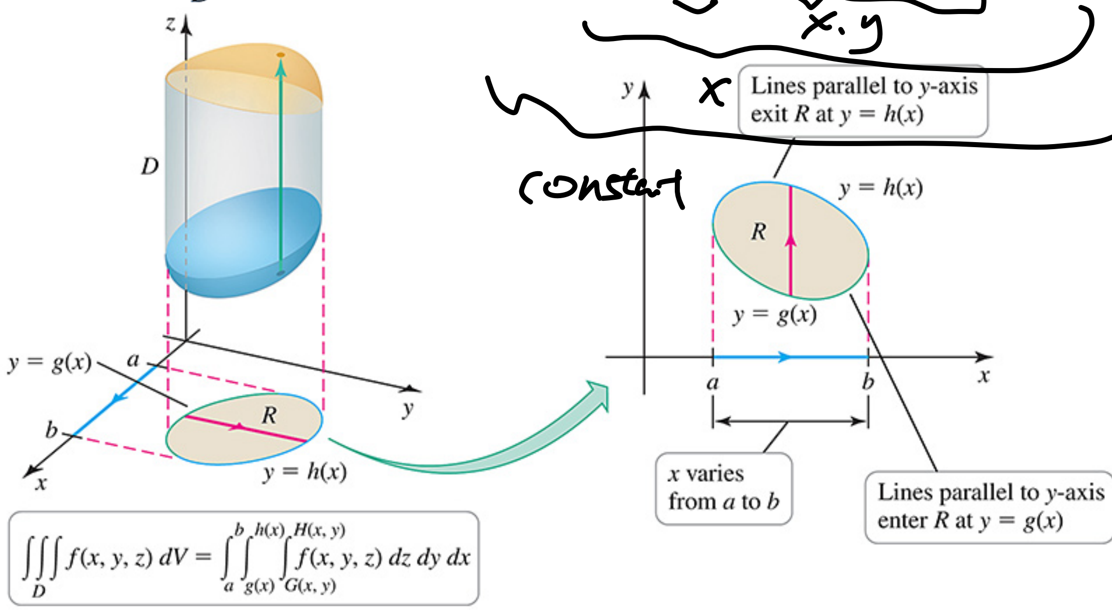
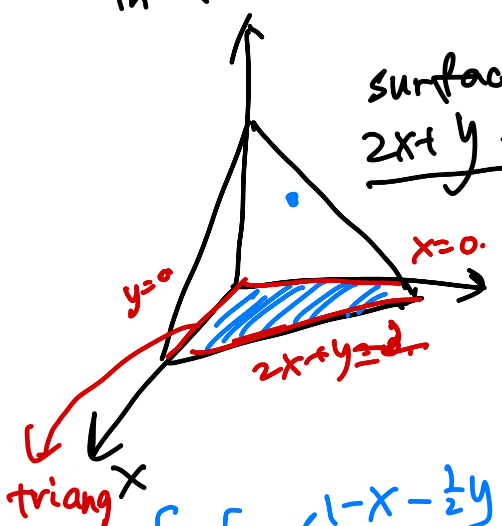


Table 16.2

Integral	Variable	Interval
Inner	z	$G(x, y) \leq z \leq H(x, y)$
Middle	y	$g(x) \leq y \leq h(x)$
Outer	x	$a \leq x \leq b$

e.g.) Find the volume of solid bound by $2x+y+2z=2$, and x-y plane, x-z plane, z-y plane

in the order $dz\,dx\,dy$.



surface top.

$2x+y+2z=2.$

First. describe the surface using variable x, y.

top $2z = 2-2x-y$

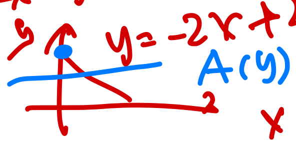
$$z = 1 - x - \frac{1}{2}y$$

bottom $z=0$.

$$\iint \int_0^{1-x-\frac{1}{2}y} 1 \, dz\,dx\,dy$$

$2x+y=2.$

$y=-2x+2.$



$x = \frac{2-y}{2}$

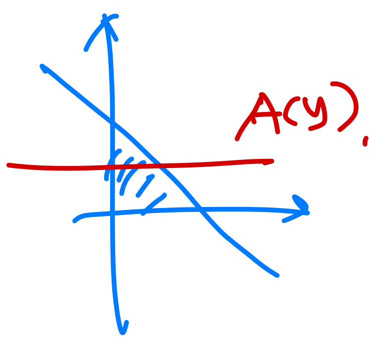
$$\int_0^2 \int_0^{1-\frac{1}{2}y} dx\,dy$$

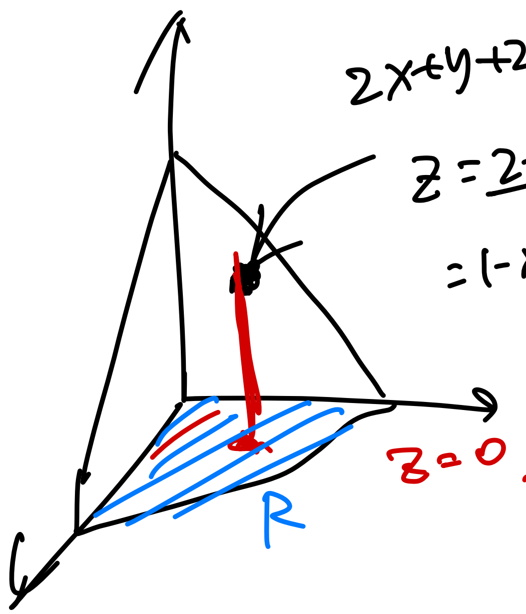
$$\int_0^2 \int_0^{1-\frac{1}{2}y} \int_0^{1-x-\frac{1}{2}y} dz\,dx\,dy.$$

* To get the volume using double integral,

$$\iint_R \text{top surface} - \text{bottom surface} \; dx\,dy$$

$$= \underline{\int_0^2 \int_0^{-\frac{1}{2}y+1}} \underbrace{\left(1-x-\tfrac{1}{2}y\right)}_{z} \underbrace{-0}_{-z} \, dx\, dy$$





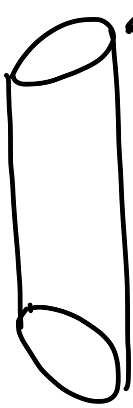
$2x+y+2z=2.$

$$z = \frac{2-2x-y}{2}$$

$$= 1-x-\tfrac{1}{2}y.$$

ex). Set up the integral which computes the volume of the wedge of the cylinder $x^2+4y^2=4$ created by planes $z=3-x$ and $z=x-3$.

$$z=f(x,y)=\underbrace{x^2+4y^2-4}_{}$$



← cylinder sliced off by $z=3-x$.

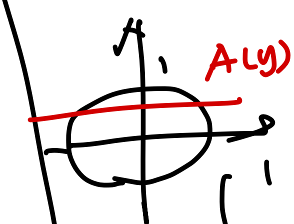
← cylinder. $x^2+4y^2=4$

← sliced off by $z=x-3$.

R ← shadow created by the cylinder obtained by letting $x^2+4y^2-4=0$ $\quad x^2+4y^2=4$.

$$\frac{1}{4}x^2+y^2=1.$$

$$\left(\frac{1}{2}x\right)^2+y^2=1$$

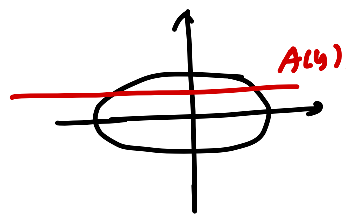


$$\int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} dx\,dy$$

$$V=\iiint 1\,dz\,dx\,dy$$

$$=\int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\int_{x-3}^{3-x} dz\,dx\,dy$$

$$\int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} (3-x)-(x-3)\,dx\,dy.$$



$$= \int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} 6-2x\,dx\,dy$$

$$= \int_{-1}^{1} 6x - x^2 \Big|_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\, dy$$

$$= \int_{-1}^{1} 12\sqrt{4-4y^2} - \left(\left(4-\cancel{4y^2}\right) - \left(4-\cancel{4y^2}\right)\right) dy$$

$$= \int_{-1}^{1} 24\sqrt{1-y^2}\, dy$$

$y=-1 \Rightarrow \theta = -\frac{\pi}{2}$

$y=\sin\theta.$ $y=1 \Rightarrow \theta=\frac{\pi}{2}.$

$dy = \cos\theta\, d\theta.$

$$=24\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta.$$

$$=24\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,\cos\theta\, d\theta.$$

$$\cos^2\theta = \frac{1+\cos2\theta}{2}.$$

$$=\frac{24}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+\cos2\theta\, d\theta = 12\int_{-1}^{1} 1+\cos2\theta\, d\theta$$

$$= 12\left(\theta + \frac{1}{2}\sin2\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}.$$

$$= 12\left(\frac{\pi}{2}+\frac{\pi}{2}\right) + 6\left(\sin\pi - \sin-\pi\right)$$

$$= \boxed{12\cdot\pi}$$

Example - Triple integral over a box

$$\int_0^2 \int_1^2 \int_0^1 y\,z\,e^x\,dx\,dz\,dy = \int_0^2 \int_1^2 yze^x \Big|_0^1 dz\,dy$$

$$= \int_0^2 \int_1^2 yz(e-1)\,dz\,dy$$

$$= \int_0^2 y\frac{z^2}{2}(e-1)\Big|_1^2 dy$$

$$= \int_0^2 y\left(2-\frac{1}{2}\right)(e-1)\,dy$$

$$= \frac{3}{2}(e-1)\int_0^2 y\,dy = \frac{3}{2}(e-1)\frac{y^2}{2}\Big|_0^2$$

$$= 3(e-1)$$

Example - Triple integral over a box

$$\int_0^2\int_1^2 \boxed{\int_0^1 y\,z\,e^x\,dx}\,dz\,dy = \int_0^2\int_1^2 \boxed{y z e^x \Big|_0^1}\,dz\,dy$$

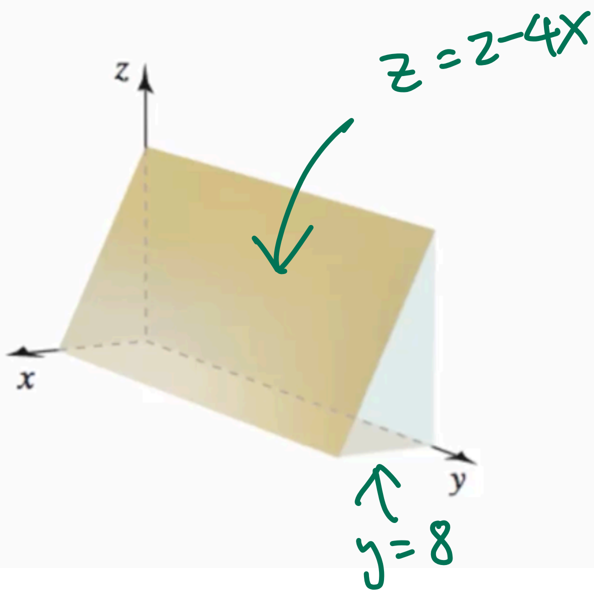
$$= \int_0^2 \boxed{\int_1^2 yz(e-1)\,dz}\,dy$$

$$= \int_0^2 \frac{1}{2}y(e-1)z^2\Big|_1^2\,dy$$

$$= \frac{3}{2}(e-1)\int_0^1 y\,dy = \frac{3}{2}(e-1)\,\frac{1}{2}y^2\Big|_0^1 = \frac{3}{4}(e-1)$$

Example - Volume of a solid

The prism in the first octant bounded by $z = 2 - 4x$ and $y = 8$.



$z=0 \Rightarrow 2-4x=0$

$2=4x$

$\frac{1}{2}=x$

$$\text{Vol} = \int_0^{\frac{1}{2}}\int_0^{8}\int_0^{2-4x} 1\,dz\,dy\,dx$$

$$= \int_0^{\frac{1}{2}}\int_0^{8} z\Big|_0^{2-4x}\,dy\,dx$$

$$= \int_0^{\frac{1}{2}}\int_0^{8} (2-4x)\,dy\,dx$$

$$= \int_0^{\frac{1}{2}} (2-4x)y\Big|_0^{8}\,dx = \int_0^{\frac{1}{2}}(2-4x)8\,dx = \int_0^{\frac{1}{2}}(16-32x)\,dx$$

$$= 16x-16x^2\Big|_0^{\frac{1}{2}} = 16\cdot\frac{1}{2} - 16\cdot\frac{1}{4} = 8-4=4.$$

Example - Evaluate the triple integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz\, dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} x z^2 \Big|_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} x(1-x^2-y^2)\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} (x - x^3 - xy^2)\, dy\, dx$$

$$= \int_0^1 \left(xy - x^3 y - \frac{xy^3}{3}\right) \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(x\sqrt{1-x^2} - x^3\sqrt{1-x^2} - \frac{x}{3}\sqrt{1-x^2}^{\,3}\right) dx$$

$$= \int_0^1 \left[(x-x^3)\sqrt{1-x^2} - \frac{x}{3}(1-x^2)\sqrt{1-x^2}\right) dx$$

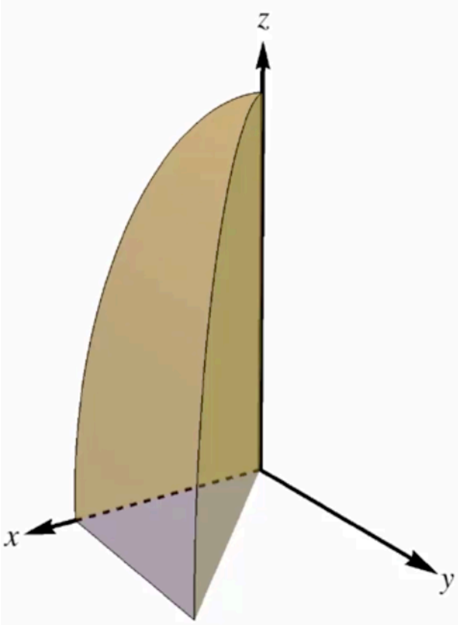
$$= \int_0^1 \left[(x-x^3) - \frac{1}{3}(x-x^3)\right]\sqrt{1-x^2}\, dx$$

$$= \int_0^1 \frac{2}{3}(x-x^3)\sqrt{1-x^2}\, dx = \int_0^1 \frac{2}{3} x(1-x^2)\sqrt{1-x^2}\, dx = \frac{2}{3}\int_0^1 x(1-x^2)^{3/2} dx$$

$$u = 1-x^2 \quad du = -2x\,dx \quad -\frac{1}{2}du = x\,dx \quad = \frac{1}{3}\int_0^1 u^{3/2} du = \frac{1}{3}\cdot\frac{2}{5}u^{5/2}\Big|_0^1 = \frac{2}{15}$$

Example - Write in the indicated order

D is the solid in the first octant bounded by the planes $y = 0$, $z = 0$, and $y = x$ and the cylinder $4x^2 + z^2 = 4$.



$$\iiint_D f(x, y, z)\, dV$$

Order: $dz\, dy\, dx$

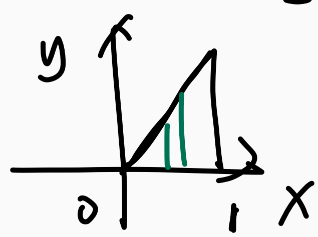
$4x^2+z^2=4 \qquad z^2=4-4x^2 \qquad z=\sqrt{4-4x^2}$

$\int_0^{\sqrt{4-4x^2}} f(x.y.z)\, dz \qquad z=0. \qquad 4x^2+0^2=4$

$x^2=1$

$x=\pm 1$

$\Rightarrow x=1$



$$\int_0^1 \int_0^x \int_0^{\sqrt{4-4x^2}} f(x.y,z)\, dz\, dy\, dx$$